

On Weakly Quasi-Conformally Symmetric Manifolds

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Abstract. The object of the present paper is to study weakly quasi-conformally symmetric Riemannian manifolds. Among others we obtain various sufficient conditions for such a manifold to be of weakly symmetric. The decomposable weakly quasi-conformally symmetric manifolds are studied and classified rigorously. The existence of a weakly quasi-conformally symmetric and decomposable weakly quasi-conformally symmetric manifolds have been ensured by several non-trivial examples.

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§1. Introduction

The notions of weakly symmetric and weakly projective symmetric manifolds were introduced by Tamássy and Binh [8] and later Binh [1] studied decomposable weakly symmetric manifolds. A non-flat Riemannian manifold (M^n, g) ($n > 2$) is called a weakly symmetric manifold if its curvature tensor R of type $(0, 4)$ satisfies the condition

$$(1.1) \quad (\nabla_X R)(Y, Z, U, V) = \alpha(X)R(Y, Z, U, V) + \beta(Y)R(X, Z, U, V) \\ + \gamma(Z)R(Y, X, U, V) + \delta(U)R(Y, Z, X, V) \\ + \sigma(V)R(Y, Z, U, X)$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where $\alpha, \beta, \gamma, \delta$ and σ are 1-forms (not simultaneously zero), $\chi(M^n)$ is the set of all smooth vector fields over the manifold and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g . The 1-forms are called the associated 1-forms of the

manifold and an n -dimensional manifold of this kind is denoted by $(WS)_n$. In 1999 U. C. De and S. Bandyopadhyay [3] established the existence of a $(WS)_n$ by an example and proved that in a $(WS)_n$, the associated 1-forms $\beta = \gamma$ and $\delta = \sigma$. Hence (1.1) reduces to the following:

$$(1.2) \quad (\nabla_X R)(Y, Z, U, V) = \alpha(X)R(Y, Z, U, V) + \beta(Y)R(X, Z, U, V) \\ + \beta(Z)R(Y, X, U, V) + \delta(U)R(Y, Z, X, V) \\ + \delta(V)R(Y, Z, U, X).$$

Also De and Bandyopadhyay [4] studied weakly conformally symmetric manifolds. In this connection it may be noted that although the definition of a $(WS)_n$ is similar to that of a generalized pseudo-symmetric manifold introduced by Chaki [2], but the defining condition of a $(WS)_n$ is little weaker than that of a generalized pseudo-symmetric manifold. That is, if in (1.1) the 1-form α is replaced by 2α and σ is replaced by α then the manifold will be a generalized pseudo-symmetric manifold [2]. In 1968 Yano and Sawaki [9] defined and studied a tensor field W on a Riemannian manifold of dimension n which includes both the conformal curvature tensor C and the concircular curvature tensor \tilde{C} as special cases. This tensor field W is known as quasi-conformal curvature tensor given by

$$(1.3) \quad W(X, Y, Z, U) = -(n-2)bC(X, Y, Z, U) \\ + [a + (n-2)b]\tilde{C}(X, Y, Z, U),$$

where a and b are arbitrary constants not simultaneously zero, C and \tilde{C} are the conformal curvature tensor and the concircular curvature tensor of type $(0, 4)$ respectively. The present paper deals with a non-quasi-conformally flat Riemannian manifold $(M^n, g)(n > 3)$ [the condition $(n > 3)$ is assumed throughout this paper as the conformal curvature tensor vanishes for $n = 3$] whose quasi-conformal curvature tensor W satisfies the condition

$$(1.4) \quad (\nabla_X W)(Y, Z, U, V) = \alpha(X)W(Y, Z, U, V) + \beta(Y)W(X, Z, U, V) \\ + \gamma(Z)W(Y, X, U, V) + \delta(U)W(Y, Z, X, V) \\ + \sigma(V)W(Y, Z, U, X),$$

where $\alpha, \beta, \gamma, \delta$ and σ are 1-forms (not simultaneously zero). Such a manifold will be called a weakly quasi-conformally symmetric manifold and denoted by $(WQCS)_n$, where the first ' W ' stands for '*weakly*' and ' QC ' stands for '*quasi-conformal curvature tensor*' as the '*weakly conformally symmetric manifold*' was denoted by $(WCS)_n$ [4]. In particular, if $a = 1$ and $b = -\frac{1}{n-2}$ then a $(WQCS)_n$ reduces to a $(WCS)_n$. The manifold $(WQCS)_n$ is introduced and studied by the first author and K. K. Baishya [7]. It is also shown in [7] that

in a $(WQCS)_n$, the 1-forms $\beta = \gamma$ and $\delta = \sigma$ and hence (1.4) reduces to the following form:

$$(1.5) \quad (\nabla_X W)(Y, Z, U, V) = \alpha(X)W(Y, Z, U, V) + \beta(Y)W(X, Z, U, V) \\ + \beta(Z)W(Y, X, U, V) + \delta(U)W(Y, Z, X, V) \\ + \delta(V)W(Y, Z, U, X),$$

where α , β and δ are 1-forms (not simultaneously zero).

Section 2 is concerned with some basic results of $(WQCS)_n$. It is shown that if in a $(WQCS)_n$ the Ricci tensor is of Codazzi [5] type then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P defined by $g(X, P) = \lambda(X)$, where r is the scalar curvature of the manifold. Also it is proved that if a $(WQCS)_n$ is of constant scalar curvature then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L_1 defined by $g(X, L_1) = \alpha(X)$. Section 3 is devoted to the decomposable $(WQCS)_n$, which is generally called the product $(WQCS)_n$ and it is shown that in such a manifold satisfying certain conditions one of the decomposition is locally symmetric and the other is quasi-conformally flat. Also we obtain some other illuminating results on a decomposable $(WQCS)_n$. Section 4 is devoted to the $(WQCS)_n$ satisfying certain conditions and obtained several interesting results for such a manifold to be a $(WS)_n$.

The last section deals with several non-trivial examples of $(WQCS)_n$ and also of decomposable $(WQCS)_n$.

§2. Some basic results of $(WQCS)_n$

In this section we deduce some basic results of a $(WQCS)_n$. The conformal curvature tensor field C of type (0, 4) and the concircular curvature tensor field \tilde{C} of type (0, 4) are respectively given by

$$C(X, Y, Z, U) = R(X, Y, Z, U) - \frac{1}{n-2}[S(Y, Z)g(X, U) - S(X, Z)g(Y, U) \\ + g(Y, Z)S(X, U) - g(X, Z)S(Y, U)] \\ + \frac{r}{(n-1)(n-2)}[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]$$

and

$$\tilde{C}(X, Y, Z, U) = R(X, Y, Z, U) - \frac{r}{n(n-1)}[g(Y, Z)g(X, U) \\ - g(X, Z)g(Y, U)],$$

for all vector fields $X, Y, Z, U \in \chi(M^n)$, where R , S , r are the Riemannian curvature tensor of type (0, 4), the Ricci tensor of type (0, 2) and the scalar

curvature respectively of the manifold. The Riemannian curvature tensor R of type $(0, 4)$ on a Riemannian manifold is defined as a quadrilinear mapping $R : \chi(M) \times \chi(M) \times \chi(M) \times \chi(M) \rightarrow C^\infty(M)$ and is given by $R(X, Y, Z, U) = g(R(X, Y)Z, U)$ for all $X, Y, Z, U \in \chi(M^n)$, where we have used the same symbol R of the curvature tensor of type $(1, 3)$ as well as of type $(0, 4)$ and $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$, ∇ being the Levi-Civita connection and $C^\infty(M)$ is the set of all smooth functions over the manifold M . The Ricci tensor field S is the covariant tensor field of degree 2 defined by $S(Y, Z) = \text{Tr}[X \rightarrow R(X, Y)Z]$ and the scalar curvature r is defined as the trace of the $(1, 1)$ Ricci tensor Q i.e., $r = \text{Tr} Q$ where $S(X, Y) = g(QX, Y)$ for all $X, Y \in \chi(M)$. Using the above expressions of Weyl conformal curvature tensor C and the concircular curvature tensor \tilde{C} in (1.3) one can easily obtain

$$(2.1) \quad \begin{aligned} W(X, Y, Z, U) = & aR(X, Y, Z, U) + b[S(Y, Z)g(X, U) \\ & - S(X, Z)g(Y, U) + g(Y, Z)S(X, U) \\ & - g(X, Z)S(Y, U)] - \frac{r}{n} \left(\frac{a}{n-1} + 2b \right) [g(Y, Z)g(X, U) \\ & - g(X, Z)g(Y, U)]. \end{aligned}$$

Let $\{e_i : i = 1, 2, \dots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then the Ricci tensor S of type $(0, 2)$ and the scalar curvature r are given by the following

$$\begin{aligned} S(X, Y) &= \sum_{i=1}^n R(e_i, X, Y, e_i) \\ \text{and } r &= \sum_{i=1}^n S(e_i, e_i) = \sum_{i=1}^n g(Qe_i, e_i). \end{aligned}$$

Again from (2.1) we can obtain

$$(2.2) \quad \begin{aligned} \sum_{i=1}^n W(e_i, Y, Z, e_i) &= \sum_{i=1}^n W(Y, e_i, e_i, Z) \\ &= \{a + (n-2)b\} [S(Y, Z) - \frac{r}{n} g(Y, Z)]. \end{aligned}$$

Differentiating (2.1) covariantly and then taking cyclic sum with respect to X, Y, Z we obtain by virtue of Bianchi identity that

$$(2.3) \quad \begin{aligned} & (\nabla_X W)(Y, Z, U, V) + (\nabla_Y W)(Z, X, U, V) + (\nabla_Z W)(X, Y, U, V) \\ &= b[\{(\nabla_X S)(Z, U) - (\nabla_Z S)(X, U)\}g(Y, V) + \{(\nabla_Y S)(X, U) \\ & - (\nabla_X S)(Y, U)\}g(Z, V) + \{(\nabla_Z S)(Y, U) - (\nabla_Y S)(Z, U)\}g(X, V) \\ & + \{(\nabla_X S)(Y, V) - (\nabla_Y S)(X, V)\}g(Z, U) + \{(\nabla_Z S)(X, V) \\ & - (\nabla_X S)(Z, V)\}g(Y, U) + \{(\nabla_Y S)(Z, V) - (\nabla_Z S)(Y, V)\}g(X, U)] \end{aligned}$$

$$\begin{aligned}
& -(\nabla_X S)(Z, V)\}g(Y, U) + \{(\nabla_Y S)(Z, V) - (\nabla_Z S)(Y, V)\}g(X, U)] \\
& -\frac{1}{n}\left(\frac{a}{n-1} + 2b\right)[dr(X)\{g(Z, U)g(Y, V) - g(Z, V)g(Y, U)\} \\
& +dr(Y)\{g(Z, V)g(X, U) - g(Z, U)g(X, V)\} \\
& +dr(Z)\{g(Y, U)g(X, V) - g(X, U)g(Y, V)\}].
\end{aligned}$$

We now suppose that in a Riemannian manifold the Ricci tensor is of Codazzi type [5]. Then we have

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) = (\nabla_Z S)(X, Y)$$

for all vector fields X, Y, Z on the manifold. This implies that

$$dr(X) = 0 \text{ for all } X.$$

Therefore (2.3) yields

$$(2.4) \quad (\nabla_X W)(Y, Z, U, V) + (\nabla_Y W)(Z, X, U, V) + (\nabla_Z W)(X, Y, U, V) = 0.$$

Hence if the Ricci tensor is of Codazzi type then in a Riemannian manifold the relation (2.4) holds. Again if a Riemannian manifold (M, g) satisfies the relation (2.4), then (2.3) yields

$$\begin{aligned}
& b[\{(\nabla_X S)(Z, U) - (\nabla_Z S)(X, U)\}g(Y, V) + \{(\nabla_Y S)(X, U) \\
& - (\nabla_X S)(Y, U)\}g(Z, V) + \{(\nabla_Z S)(Y, U) - (\nabla_Y S)(Z, U)\}g(X, V) \\
& + \{(\nabla_X S)(Y, V) - (\nabla_Y S)(X, V)\}g(Z, U) + \{(\nabla_Z S)(X, V) \\
& - (\nabla_X S)(Z, V)\}g(Y, U) + \{(\nabla_Y S)(Z, V) - (\nabla_Z S)(Y, V)\}g(X, U)] \\
& -\frac{1}{n}\left(\frac{a}{n-1} + 2b\right)[dr(X)\{g(Z, U)g(Y, V) - g(Z, V)g(Y, U)\} \\
& +dr(Y)\{g(Z, V)g(X, U) - g(Z, U)g(X, V)\} \\
& +dr(Z)\{g(Y, U)g(X, V) - g(X, U)g(Y, V)\}] = 0.
\end{aligned}$$

Setting $Y = V = e_i$ in the above relation and then taking summation over i , $1 \leq i \leq n$ we get

$$\begin{aligned}
& (n-3)b[(\nabla_X S)(Z, U) - (\nabla_Z S)(X, U)] - \left\{\frac{(n-2)a}{n(n-1)}\right. \\
& \left. + \frac{(3n-8)b}{2n}\right\}[dr(X)g(Z, U) - dr(Z)g(X, U)] = 0,
\end{aligned}$$

which yields on contraction over Z and U that $dr(X) = 0$ for all X provided $a + (n-2)b \neq 0$ and consequently the last relation reduces to

$$\begin{aligned}
& (\nabla_X S)(Z, U) = (\nabla_Z S)(X, U) \\
& \text{for all } X, Z, U \in \chi(M) \text{ provided } b \neq 0.
\end{aligned}$$

Hence the Ricci tensor is of Codazzi type. Thus we can state the following:

Proposition 2.1. *In a Riemannian manifold (M^n, g) with $b \neq 0$ and $a + (n - 2)b \neq 0$, the Ricci tensor is of Codazzi type if and only if the relation (2.4) holds.*

In view of (1.5), the relation (2.4) reduces to

$$(2.5) \quad \lambda(X)W(Y, Z, U, V) + \lambda(Y)W(Z, X, U, V) + \lambda(Z)W(X, Y, U, V) = 0,$$

where $\lambda(X) = \alpha(X) - 2\beta(X)$ for all X . By virtue of (2.1), (2.5) takes the form

$$(2.6) \quad \begin{aligned} & a[\lambda(X)R(Y, Z, U, V) + \lambda(Y)R(Z, X, U, V) + \lambda(Z)R(X, Y, U, V)] \\ & + b[\lambda(X)\{S(Z, U)g(Y, V) - S(Y, U)g(Z, V) + S(Y, V)g(Z, U) \\ & - S(Z, V)g(Y, U)\} + \lambda(Y)\{S(X, U)g(Z, V) - S(Z, U)g(X, V) \\ & + S(Z, V)g(X, U) - S(X, V)g(Z, U)\} + \lambda(Z)\{S(Y, U)g(X, V) \\ & - S(X, U)g(Y, V) + S(X, V)g(Y, U) - S(Y, V)g(X, U)\}] \\ & - \frac{r}{n}(\frac{a}{n-1} + 2b)[\lambda(X)\{g(Z, U)g(Y, V) - g(Y, U)g(Z, V)\} \\ & + \lambda(Y)\{g(X, U)g(Z, V) - g(Z, U)g(X, V)\} \\ & + \lambda(Z)\{g(Y, U)g(X, V) - g(X, U)g(Y, V)\}] = 0. \end{aligned}$$

Setting $Y = V = e_i$ in (2.6) and taking summation over i , $1 \leq i \leq n$, we get

$$(2.7) \quad \begin{aligned} & \{a + (n - 3)b\}[\lambda(X)S(Z, U) - \lambda(Z)S(X, U)] + a\lambda(R(Z, X)U) \\ & + b[\lambda(QZ)g(X, U) - \lambda(QX)g(Z, U)] - \frac{r}{n}\{\frac{(n-2)a}{n-1} \\ & + (n-4)b\}[\lambda(X)g(Z, U) - \lambda(Z)g(X, U)] = 0. \end{aligned}$$

Again putting $X = U = e_i$ in (2.7) and taking summation over i , $1 \leq i \leq n$, we obtain

$$\begin{aligned} & \{a + (n - 2)b\}[\lambda(QZ) - \frac{r}{n}\lambda(Z)] = 0, \quad \text{which yields} \\ & S(Z, P) = \frac{r}{n}g(Z, P), \end{aligned}$$

provided that $a + (n - 2)b \neq 0$ where $\lambda(X) = \alpha(X) - 2\beta(X)$ and $g(X, P) = \lambda(X)$. This leads to the following:

Proposition 2.2. *If in a $(WQCS)_n$ the Ricci tensor is of Codazzi type then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P , defined by $g(X, P) = \lambda(X)$, provided that $a + (n - 2)b \neq 0$.*

Next in view of (1.5), the relation (2.3) takes the form

$$\begin{aligned}
(2.8) \quad & b[\{(\nabla_X S)(Z, U) - (\nabla_Z S)(X, U)\}g(Y, V) + \{(\nabla_Y S)(X, U) \\
& - (\nabla_X S)(Y, U)\}g(Z, V) + \{(\nabla_Z S)(Y, U) - (\nabla_Y S)(Z, U)\}g(X, V) \\
& + \{(\nabla_X S)(Y, V) - (\nabla_Y S)(X, V)\}g(Z, U) + \{(\nabla_Z S)(X, V) \\
& - (\nabla_X S)(Z, V)\}g(Y, U) + \{(\nabla_Y S)(Z, V) - (\nabla_Z S)(Y, V)\}g(X, U)] \\
& - \frac{1}{n} \left(\frac{a}{n-1} + 2b \right) [dr(X)\{g(Z, U)g(Y, V) - g(Z, V)g(Y, U)\} \\
& + dr(Y)\{g(Z, V)g(X, U) - g(Z, U)g(X, V)\} \\
& + dr(Z)\{g(Y, U)g(X, V) - g(X, U)g(Y, V)\}] \\
& = \lambda(X)W(Y, Z, U, V) + \lambda(Y)W(Z, X, U, V) + \lambda(Z)W(X, Y, U, V),
\end{aligned}$$

where $\lambda(X) = \alpha(X) - 2\beta(X)$ for all X . Setting $Y = V = e_i$ in (2.8) and taking summation over i , $1 \leq i \leq n$, we obtain by virtue of (2.1) and (2.2) that

$$\begin{aligned}
(2.9) \quad & (n-3)b[(\nabla_X S)(Z, U) - (\nabla_Z S)(X, U)] \\
& - \left[\frac{a(n-2)}{n(n-1)} - \frac{b(3n-8)}{2n} \right] [dr(X)g(Z, U) - dr(Z)g(X, U)] \\
& = [a + (n-3)b][\lambda(X)S(Z, U) - \lambda(Z)S(X, U)] \\
& + a\lambda(R(Z, X)U) + b[\lambda(QZ)g(X, U) - \lambda(QX)g(Z, U)] \\
& - \frac{r}{n} \left[\frac{(n-2)a}{n-1} + (n-4)b \right] [\lambda(X)g(Z, U) - \lambda(Z)g(X, U)].
\end{aligned}$$

Putting $X = U = e_i$ in (2.9) and taking summation over i , $1 \leq i \leq n$, we get

$$(2.10) \quad \frac{n-2}{2n} dr(Z) = \lambda(QZ) - \frac{r}{n} \lambda(Z) \quad \text{for } a + (n-2)b \neq 0.$$

If the manifold under consideration is of constant scalar curvature then (2.10) yields

$$(2.11) \quad \lambda(QZ) = \frac{r}{n} \lambda(Z) \quad \text{for } a + (n-2)b \neq 0.$$

If P is the vector field associated with λ such that $g(X, P) = \lambda(X) = \alpha(X) - 2\beta(X)$ then (2.11) can be written as

$$(2.12) \quad S(Z, P) = \frac{r}{n} g(Z, P) \quad \text{for } a + (n-2)b \neq 0.$$

Thus we can state the following:

Proposition 2.3. *If a $(WQCS)_n$ is of constant scalar curvature with $a + (n-2)b \neq 0$ then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P , defined by $g(X, P) = \lambda(X)$ for all X .*

Now using (2.1) in (1.5) we obtain

$$\begin{aligned}
(2.13) \quad & a(\nabla_X R)(Y, Z, U, V) + b[(\nabla_X S)(Z, U)g(Y, V) - (\nabla_X S)(Y, U)g(Z, V) \\
& + (\nabla_X S)(Y, V)g(Z, U) - (\nabla_X S)(Z, V)g(Y, U)] \\
& - \frac{1}{n}dr(X)\left(\frac{a}{n-1} + 2b\right)[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)] \\
& = a[\alpha(X)R(Y, Z, U, V) + \beta(Y)R(X, Z, U, V) + \beta(Z)R(Y, X, U, V) \\
& + \delta(U)R(Y, Z, X, V) + \delta(V)R(Y, Z, U, X)] + b[\alpha(X)\{S(Z, U)g(Y, V) \\
& - S(Y, U)g(Z, V) + S(Y, V)g(Z, U) - S(Z, V)g(Y, U)\} \\
& + \beta(Y)\{S(Z, U)g(X, V) - S(X, U)g(Z, V) + S(X, V)g(Z, U) \\
& - S(Z, V)g(X, U)\} + \beta(Z)\{S(X, U)g(Y, V) - S(Y, U)g(X, V) \\
& + S(Y, V)g(X, U) - S(X, V)g(Y, U)\} + \delta(U)\{S(Z, X)g(Y, V) \\
& - S(X, Y)g(Z, V) + S(Y, V)g(Z, X) - S(Z, V)g(X, Y)\} \\
& + \delta(V)\{S(Z, U)g(X, Y) - S(Y, U)g(Z, X) + S(X, Y)g(Z, U) \\
& - S(Z, X)g(Y, U)\}] - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[\alpha(X)\{g(Z, U)g(Y, V) \\
& - g(Y, U)g(Z, V)\} + \beta(Y)\{g(Z, U)g(X, V) - g(X, U)g(Z, V)\} \\
& + \beta(Z)\{g(X, U)g(Y, V) - g(Y, U)g(X, V)\} \\
& + \delta(U)\{g(Z, X)g(Y, V) - g(X, Y)g(Z, V)\} \\
& + \delta(V)\{g(Z, U)g(X, Y) - g(Y, U)g(Z, X)\}].
\end{aligned}$$

Setting $Y = V = e_i$ in (2.13) and taking summation over i , $1 \leq i \leq n$, we get

$$\begin{aligned}
(2.14) \quad & \{a + (n-2)b\}[(\nabla_X S)(Z, U) - \frac{1}{n}dr(X)g(Z, U)] \\
& = \{a + (n-2)b\}[\alpha(X)\{S(Z, U) - \frac{r}{n}g(Z, U)\} \\
& + \beta(Z)\{S(X, U) - \frac{r}{n}g(X, U)\} \\
& + \delta(U)\{S(Z, X) - \frac{r}{n}g(Z, X)\}] + a[\beta(R(X, Z)U) \\
& + \delta(R(X, U)Z)] + b[\beta(X)S(Z, U) - \beta(Z)S(X, U) \\
& + \delta(X)S(Z, U) - \delta(U)S(Z, X) \\
& + \beta(QX)g(Z, U) - \beta(QZ)g(X, U) + \delta(QX)g(Z, U) \\
& - \delta(QU)g(Z, X)] - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[\beta(X)g(Z, U) \\
& - \beta(Z)g(X, U) + \delta(X)g(Z, U) - \delta(U)g(Z, X)].
\end{aligned}$$

Again contracting (2.14) over Z and U we obtain

$$(2.15) \quad \beta(QX) + \delta(QX) = \frac{r}{n}[\beta(X) + \delta(X)], \quad \text{for } a + (n-2)b \neq 0.$$

Also contracting (2.14) over X and U we have

$$(2.16) \quad \begin{aligned} \frac{n-2}{2n}dr(Z) &= \alpha(QZ) - \beta(QZ) + \delta(QZ) \\ &\quad - \frac{r}{n}[\alpha(Z) - \beta(Z) + \delta(Z)], \end{aligned}$$

for $a + (n-2)b \neq 0$. Furthermore, contracting (2.14) over X and Z we obtain

$$(2.17) \quad \begin{aligned} \frac{n-2}{2n}dr(U) &= \alpha(QU) + \beta(QU) - \delta(QU) \\ &\quad - \frac{r}{n}[\alpha(U) + \beta(U) - \delta(U)], \end{aligned}$$

provided that $a + (n-2)b \neq 0$. Replacing U by Z in (2.17) yields

$$(2.18) \quad \begin{aligned} \frac{n-2}{2n}dr(Z) &= \alpha(QZ) + \beta(QZ) - \delta(QZ) \\ &\quad - \frac{r}{n}[\alpha(Z) + \beta(Z) - \delta(Z)]. \end{aligned}$$

From (2.16) and (2.18) it follows that

$$(2.19) \quad \beta(QZ) - \delta(QZ) = \frac{r}{n}[\beta(Z) - \delta(Z)], \text{ for } a + (n-2)b \neq 0.$$

In view of (2.15) and (2.19), we obtain

$$(2.20) \quad \beta(QZ) = \frac{r}{n}\beta(Z)$$

and

$$(2.21) \quad \delta(QZ) = \frac{r}{n}\delta(Z), \text{ for } a + (n-2)b \neq 0.$$

This leads to the following:

Proposition 2.4. *In a $(WQCS)_n$ with $a + (n-2)b \neq 0$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvectors L_2 and L_3 defined by $g(X, L_2) = \beta(X)$ and $g(X, L_3) = \delta(X)$ respectively, for all X .*

Using (2.20) and (2.21) in (2.18) we get

$$(2.22) \quad \frac{n-2}{2n}dr(Z) = \alpha(QZ) - \frac{r}{n}\alpha(Z), \text{ for } a + (n-2)b \neq 0.$$

If the manifold is of constant scalar curvature then (2.22) yields

$$(2.23) \quad \alpha(QZ) = \frac{r}{n}\alpha(Z), \text{ for } a + (n-2)b \neq 0.$$

Thus we can state the following:

Proposition 2.5. *If a $(WQCS)_n$ is of constant scalar curvature with $a + (n - 2)b \neq 0$, then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L_1 defined by $g(X, L_1) = \alpha(X)$ for all X .*

Using (2.20) and (2.21) in (2.14) we obtain

$$\begin{aligned}
 (2.24) \quad & \{a + (n - 2)b\}[(\nabla_X S)(Z, U) - \frac{1}{n}dr(X)g(Z, U)] \\
 &= \{a + (n - 2)b\}[\alpha(X)\{S(Z, U) - \frac{r}{n}g(Z, U)\} + \beta(Z)\{S(X, U) \\
 &\quad - \frac{r}{n}g(X, U)\} + \delta(U)\{S(Z, X) - \frac{r}{n}g(Z, X)\}] \\
 &\quad + a[\beta(R(X, Z)U) + \delta(R(X, U)Z)] + b[\beta(X)S(Z, U) \\
 &\quad - \beta(Z)S(X, U) + \delta(X)S(Z, U) - \delta(U)S(Z, X)] \\
 &\quad - \frac{r}{n}(\frac{a}{n-1} + b)[\beta(X)g(Z, U) - \beta(Z)g(X, U) \\
 &\quad + \delta(X)g(Z, U) - \delta(U)g(Z, X)].
 \end{aligned}$$

The above results will be used in the later sections.

§3. Decomposable $(WQCS)_n$

A Riemannian manifold (M^n, g) is said to be decomposable or product manifold [6] if it can be expressed as $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq n - 2$.

Let (M^n, g) be a Riemannian manifold such that $M^n = M_1^p \times M_2^{n-p}$ ($2 \leq p \leq n - 2$). We assume that M is a weakly quasi-conformally symmetric manifold, that is, for $X, Y, Z, U, V \in \chi(M)$

$$\begin{aligned}
 (\nabla_X W)(Y, Z, U, V) &= \alpha(X)W(Y, Z, U, V) + \beta(Y)W(X, Z, U, V) \\
 &\quad + \beta(Z)W(Y, X, U, V) + \delta(U)W(Y, Z, X, V) \\
 &\quad + \delta(V)W(Y, Z, U, X),
 \end{aligned}$$

where α , β and δ are (not simultaneously zero) 1-forms on M . Then we find

$$\begin{aligned}
 (3.1) \quad (\nabla_{\bar{X}} W)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) &= \alpha(\bar{X})W(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) + \beta(\bar{Y})W(\bar{X}, \bar{Z}, \bar{U}, \bar{V}) \\
 &\quad + \beta(\bar{Z})W(\bar{Y}, \bar{X}, \bar{U}, \bar{V}) + \delta(\bar{U})W(\bar{Y}, \bar{Z}, \bar{X}, \bar{V}) \\
 &\quad + \delta(\bar{V})W(\bar{Y}, \bar{Z}, \bar{U}, \bar{X}),
 \end{aligned}$$

$$(3.2) \quad \alpha(\bar{X})^* W(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0,$$

$$(3.3) \quad \beta(\bar{Y})^* W(\bar{X}, \bar{Z}, \bar{U}, \bar{V}) = 0,$$

$$(3.4) \quad \delta(\bar{U})^* W(\bar{Y}, \bar{Z}, \bar{X}, \bar{V}) = 0,$$

$$(3.5) \quad \beta(\bar{Z})W(\bar{X}^*, \bar{Y}^*, \bar{U}, \bar{V}) + \delta(\bar{U})W(\bar{X}^*, \bar{V}, \bar{Z}, \bar{Y}^*) - \delta(\bar{V})W(\bar{X}^*, \bar{U}, \bar{Z}, \bar{Y}^*) = 0,$$

$$(3.6) \quad \beta(\bar{Y})W(\bar{X}^*, \bar{Z}, \bar{V}, \bar{U}) - \beta(\bar{Z})W(\bar{X}^*, \bar{Y}, \bar{V}, \bar{U}) + \delta(\bar{V})W(\bar{X}^*, \bar{U}, \bar{Y}, \bar{Z}) = 0,$$

$$(3.7) \quad (\nabla_{\bar{X}} W)(\bar{Y}^*, \bar{Z}, \bar{U}, \bar{V}) = \alpha(\bar{X})W(\bar{Y}^*, \bar{Z}, \bar{U}, \bar{V}) + \beta(\bar{Z})W(\bar{Y}^*, \bar{X}, \bar{U}, \bar{V}) \\ + \delta(\bar{U})W(\bar{Y}^*, \bar{Z}, \bar{X}, \bar{V}),$$

$$(3.8) \quad (\nabla_{\bar{X}}^* W)(\bar{Y}^*, \bar{Z}, \bar{U}, \bar{V}) = \alpha(\bar{X}^*)W(\bar{Y}^*, \bar{Z}, \bar{U}, \bar{V}) + \beta(\bar{Y}^*)W(\bar{X}^*, \bar{Z}, \bar{U}, \bar{V}) \\ + \delta(\bar{V})W(\bar{Y}^*, \bar{Z}, \bar{U}, \bar{X}^*),$$

$$(3.9) \quad \beta(\bar{Z}^*)W(\bar{X}, \bar{Y}, \bar{U}^*, \bar{V}^*) + \delta(\bar{U}^*)W(\bar{Z}^*, \bar{Y}, \bar{X}, \bar{V}^*) - \delta(\bar{V}^*)W(\bar{Z}^*, \bar{Y}, \bar{X}, \bar{U}^*) = 0,$$

$$(3.10) \quad \beta(\bar{Y}^*)W(\bar{Z}^*, \bar{X}, \bar{U}, \bar{V}^*) - \beta(\bar{Z}^*)W(\bar{Y}^*, \bar{X}, \bar{U}, \bar{V}^*) + \delta(\bar{V}^*)W(\bar{Y}^*, \bar{Z}^*, \bar{X}, \bar{U}) = 0,$$

$$(3.11) \quad \alpha(\bar{X})W(\bar{Y}^*, \bar{Z}, \bar{U}, \bar{V}) = 0,$$

$$(3.12) \quad \beta(\bar{Y})W(\bar{X}^*, \bar{Z}, \bar{U}, \bar{V}) = 0,$$

$$(3.13) \quad \delta(\bar{U})W(\bar{Y}^*, \bar{Z}, \bar{X}, \bar{V}) = 0,$$

$$(3.14) \quad (\nabla_{\bar{X}}^* W)(\bar{Y}^*, \bar{Z}, \bar{U}, \bar{V}) = \alpha(\bar{X}^*)W(\bar{Y}^*, \bar{Z}, \bar{U}, \bar{V}) + \beta(\bar{Y}^*)W(\bar{X}^*, \bar{Z}, \bar{U}, \bar{V}) \\ + \beta(\bar{Z}^*)W(\bar{Y}^*, \bar{X}, \bar{U}, \bar{V}) + \delta(\bar{U}^*)W(\bar{Y}^*, \bar{Z}, \bar{X}, \bar{V}) \\ + \delta(\bar{V}^*)W(\bar{Y}^*, \bar{Z}, \bar{U}, \bar{X})$$

for $\bar{X}, \bar{Y}, \bar{Z}, \bar{U}, \bar{V} \in \chi(M_1)$ and $\bar{X}^*, \bar{Y}^*, \bar{Z}^*, \bar{U}^*, \bar{V}^* \in \chi(M_2)$. From (3.2)-(3.4), we have two cases, namely,

(1) $\alpha = 0, \beta = 0, \delta = 0$ on M_2 ,

(2) M_1 is a quasi-conformally flat.

At first, we consider the case (1). Then from (3.8) it follows that

$$(\nabla_{\bar{X}}^* W)(\bar{Y}^*, \bar{Z}, \bar{U}, \bar{V}) = 0, \quad \text{which implies that}$$

$$(3.15) \quad b(\nabla_{\bar{X}}^* S)(\bar{Y}^*, \bar{V}) = \frac{\bar{X}^* r}{n} \left(\frac{a}{n-1} + 2b \right) g(\bar{Y}^*, \bar{V}).$$

Also from (3.14), we obtain

$$(\nabla_{\bar{X}}^* W)(\bar{Y}^*, \bar{Z}, \bar{U}^*, \bar{V}) = 0, \quad \text{that is,}$$

$$\begin{aligned}
(3.16) \quad & a(\nabla_X^* R)(Y^*, Z^*, U^*, V^*) \\
& + b\{(\nabla_X^* S)(Z^*, U^*)g(Y^*, V^*) - (\nabla_X^* S)(Y^*, U^*)g(Z^*, V^*) \\
& + g(Z^*, U^*)(\nabla_X^* S)(Y^*, V^*) - g(Y^*, U^*)(\nabla_X^* S)(Z^*, V^*)\} \\
& - \frac{X^* r}{n} \left(\frac{a}{n-1} + 2b \right) \{g(Z^*, U^*)g(Y^*, V^*) - g(Y^*, U^*)g(Z^*, V^*)\} = 0,
\end{aligned}$$

which yields that

$$\begin{aligned}
(3.17) \quad & \{a + (n - p - 2)b\}(\nabla_X^* S)(Y^*, V^*) \\
& = \frac{X^* r}{n} \left\{ \frac{n - p - 1}{n - 1} a + (n - 2p - 2)b \right\} g(Y^*, V^*),
\end{aligned}$$

where we denote the scalar curvature on M_2 by $\overset{*}{r}$. It is easy to see from (3.15), (3.17) and $\overset{*}{X}r = \overset{*}{X}r$ that

$$\{a + (n - 1)b\}\{a + (n - 2)b\} \overset{*}{X}r = 0.$$

Thus we have the following three cases:

$$(1-1) \quad a + (n - 1)b = 0;$$

$$(1-2) \quad a + (n - 2)b = 0;$$

$$(1-3) \quad \overset{*}{X}r = 0.$$

In the case of (1-1), we find from (3.15) and $b \neq 0$

$$(\nabla_X^* S)(Y^*, V^*) = \frac{\overset{*}{X}r}{n} g(Y^*, V^*),$$

$$\text{which implies that } \overset{*}{X}r = 0.$$

Thus we have $(\nabla_X^* S)(Y^*, V^*) = 0$. Similarly, if the case (1-2) holds, then we get $(\nabla_X^* S)(Y^*, V^*) = 0$. By virtue of (3.15) and (3.17), when (1-3) holds, we have $(\nabla_X^* S)(Y^*, V^*) = 0$ if $a \neq 0$ or $b \neq 0$. Moreover, from (3.16) we find

$$\begin{aligned}
& (\nabla_X^* R)(Y^*, Z^*, U^*, V^*) = 0 \\
& \text{if } a \neq 0.
\end{aligned}$$

Secondly, we discuss the case of (2). From $W = 0$ on M_1 , we find

$$\begin{aligned}
(3.18) \quad & aR(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) + b[S(\bar{Y}, \bar{Z})g(\bar{X}, \bar{U}) - S(\bar{X}, \bar{Z})g(\bar{Y}, \bar{U}) \\
& + g(\bar{Y}, \bar{Z})S(\bar{X}, \bar{U}) - g(\bar{X}, \bar{Z})S(\bar{Y}, \bar{U})] \\
& - \frac{\bar{r}}{n} \left(\frac{a}{n-1} + 2b \right) \{g(\bar{Y}, \bar{Z})g(\bar{X}, \bar{U}) - g(\bar{X}, \bar{Z})g(\bar{Y}, \bar{U})\} = 0,
\end{aligned}$$

which implies that

$$(3.19) \quad \{a + (p-2)b\}S(\bar{Y}, \bar{Z}) + \{b\bar{r} - \frac{(p-1)r}{n}(\frac{a}{n-1} + 2b)\}g(\bar{Y}, \bar{Z}) = 0,$$

where \bar{r} is the scalar curvature on M_1 . Thus we find

$$(3.20) \quad b\bar{r} - \frac{(p-1)r}{n}(\frac{a}{n-1} + 2b) = -\frac{\bar{r}}{p}\{a + (p-2)b\}.$$

Using (3.20) in (3.19) we obtain

$$\{a + (p-2)b\}\{S(\bar{Y}, \bar{Z}) - \frac{\bar{r}}{p}g(\bar{Y}, \bar{Z})\} = 0.$$

Therefore we can consider the following two cases:

$$(2-1) \quad a + (p-2)b = 0;$$

$$(2-2) \quad a + (p-2)b \neq 0.$$

In the case of (2-1), we get from (3.18), (3.20) and $b \neq 0$

$$(3.21) \quad \begin{aligned} & (p-2)R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) - \{S(\bar{Y}, \bar{Z})g(\bar{X}, \bar{U}) - S(\bar{X}, \bar{Z})g(\bar{Y}, \bar{U}) \\ & + g(\bar{Y}, \bar{Z})S(\bar{X}, \bar{U}) - g(\bar{X}, \bar{Z})S(\bar{Y}, \bar{U})\} \\ & + \frac{\bar{r}}{p-1}\{g(\bar{Y}, \bar{Z})g(\bar{X}, \bar{U}) - g(\bar{X}, \bar{Z})g(\bar{Y}, \bar{U})\} = 0. \end{aligned}$$

Thus M_1 is conformally flat if $p \neq 2$. Also, in the case of (2-2), equation (3.18) is rewritten as follows:

$$(3.22) \quad R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) = \frac{\bar{r}}{p(p-1)}\{g(\bar{Y}, \bar{Z})g(\bar{X}, \bar{U}) - g(\bar{X}, \bar{Z})g(\bar{Y}, \bar{U})\},$$

if $a \neq 0$. Hence we have

Theorem 3.1. *Let (M^n, g) be a Riemannian manifold such that $M = M_1^p \times M_2^{n-p}$ ($2 \leq p \leq n-2$). If M is a $(WQCS)_n$, then we get*

(1) *in the case of $\alpha = 0$, $\beta = 0$, $\delta = 0$ on M_2 , M_2 is a locally symmetric manifold for $a \neq 0$,*

(2) *when M_1 is a quasi-conformally flat,*

(i) *if $a + (p-2)b = 0$ and $p \geq 3$, then M_1 is conformally flat,*

(ii) *if $a \neq 0$, $a + (p-2)b \neq 0$ and $p \geq 3$, then M_1 is a manifold of constant curvature.*

Similarly we have from (3.11)–(3.13)

Theorem 3.2. *Let (M^n, g) be a Riemannian manifold such that $M = M_1^p \times M_2^{n-p}$ ($2 \leq p \leq n-2$). If M is a $(WQCS)_n$, then we get*

(1) *in the case of $\alpha = 0$, $\beta = 0$, $\delta = 0$ on M_1 , M_1 is a locally symmetric*

manifold for $a \neq 0$,

(2) when M_2 is a quasi-conformally flat,

(i) if $a + (p-2)b = 0$ and $p \leq n-3$, then M_2 is conformally flat,

(ii) if $a \neq 0, a + (p-2)b \neq 0$ and $p \leq n-3$, then M_2 is of constant curvature.

Next, we consider the contraction with respect to \bar{X}^* and \bar{U}^* in (3.6) and obtain

$$\begin{aligned} & \beta(\bar{Y})[b\{\bar{r}^* g(\bar{Z}, \bar{V}) + (n-p)S(\bar{Z}, \bar{V})\} - \frac{(n-p)r}{n}(\frac{a}{n-1} + 2b)g(\bar{Z}, \bar{V})] \\ & - \beta(\bar{Z})[b\{\bar{r}^* g(\bar{Y}, \bar{V}) + (n-p)S(\bar{Y}, \bar{V})\} - \frac{(n-p)r}{n}(\frac{a}{n-1} + 2b)g(\bar{Y}, \bar{V})] \\ & = 0, \end{aligned}$$

which yields that

$$(3.23) \quad b(n-p)\beta(Q\bar{Y}) = -\frac{r_1}{n}\beta(\bar{Y}),$$

where we put

$$r_1 = (n-p)\{\frac{p-1}{n-1}a - (n-2p+2)b\}\bar{r} + (p-1)\{\frac{n-p}{n-1}a + (n-2p)b\}\bar{r}^*.$$

Similarly, we have from (3.5)

$$(3.24) \quad b(n-p)\delta(Q\bar{U}) = -\frac{r_1}{n}\delta(\bar{U}).$$

If $b = 0$, that is, $W = a\tilde{C}$ on M , then from (3.23) and (3.24) we get $r\beta(\bar{Y}) = 0$ and $r\delta(\bar{U}) = 0$. Thus we can consider the two cases:

(3) $r = 0$,

(4) $r \neq 0$, namely, $\beta = 0, \delta = 0$ on M_1 .

If $r = 0$, then M is a weakly symmetric manifold. When the case of (4) holds, we obtain from (3.7) that

$$(3.25) \quad \alpha(\bar{X}) = -\bar{X} \log |r|.$$

It is clear from (3.1) that

$$(3.26) \quad (\nabla_{\bar{X}}\tilde{C})(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = \alpha(\bar{X})\tilde{C}(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}).$$

Hence we can state the following:

Theorem 3.3. *Let (M^n, g) be a Riemannian manifold such that $M = M_1^p \times M_2^{n-p}$ ($2 \leq p \leq n-2$). If M is a $(WQCS)_n$, then we get*

(1) if $b \neq 0$, then we find

$$\begin{aligned} \beta(Q\cdot) &= -\frac{r_1}{bn(n-p)}\beta(\cdot) \\ \text{and} \quad \delta(Q\cdot) &= -\frac{r_1}{bn(n-p)}\delta(\cdot) \quad \text{on } M_1, \end{aligned}$$

(2) in the case of $b = 0$,

(i) if $r = 0$, then M is a weakly symmetric manifold,

(ii) if $r \neq 0$, then $\alpha(\bar{X}) = -\bar{X} \log |r|$ and $\nabla_{\bar{X}} \tilde{C} = \alpha(\bar{X}) \tilde{C}$ on M_1 for $\bar{X} \in \chi(M_1)$. Especially, if r is a non-zero constant, then the concircular curvature tensor field is parallel on M_1 .

Similarly, from (3.9) and (3.10) we can state the following

Theorem 3.4. Let (M^n, g) be a Riemannian manifold such that $M = M_1^p \times M_2^{n-p}$ ($2 \leq p \leq n-2$). If M is a $(WQCS)_n$, then we get

(1) if $b \neq 0$, then we find

$$\begin{aligned} \beta(Q \cdot) &= -\frac{r_2}{bnp} \beta(\cdot) \\ \text{and} \quad \delta(Q \cdot) &= -\frac{r_2}{bnp} \delta(\cdot) \quad \text{on } M_2, \end{aligned}$$

where we put

$$r_2 = (n-p-1) \left\{ \frac{pa}{n-1} - (n-2p)b \right\} \bar{r} + p \left\{ \frac{n-p-1}{n-1} a + (n-2p-2)b \right\} \bar{r}^*,$$

(2) in the case of $b = 0$,

(i) if $r = 0$, then M is a weakly symmetric manifold,

(ii) if $r \neq 0$, then $\alpha(\bar{X}) = -\bar{X} \log |r|$ and $\nabla_{\bar{X}}^* \tilde{C} = \alpha(\bar{X}) \tilde{C}$ on M_2 for $\bar{X} \in \chi(M_2)$. Especially, if r is a non-zero constant, then the concircular curvature tensor field is parallel on M_2 .

§4. $(WQCS)_n$ satisfying certain conditions

Definition 4.1. The Ricci tensor of a Riemannian manifold is said to be *cyclic parallel* if it satisfies the following condition:

$$(4.1) \quad (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$$

for all vector fields X, Y, Z on the manifold i.e., the Ricci tensor S of a Riemannian manifold is cyclic parallel if the cyclic sum of the covariant derivative of S vanishes.

From (4.1) it follows that in such a manifold the scalar curvature r is a constant.

We now consider a $(WQCS)_n$ satisfying the relation (4.1). Taking cyclic sum

with respect to X, Z, U in (2.24) we obtain by virtue of (4.1) and Bianchi identity that

$$(4.2) \quad \begin{aligned} & \{\alpha(X) + \beta(X) + \delta(X)\} [S(Z, U) - \frac{r}{n}g(Z, U)] \\ & + \{\alpha(Z) + \beta(Z) + \delta(Z)\} [S(X, U) - \frac{r}{n}g(X, U)] \\ & + \{\alpha(U) + \beta(U) + \delta(U)\} [S(Z, X) - \frac{r}{n}g(Z, X)] = 0 \end{aligned}$$

for $a + (n - 2)b \neq 0$ and $\alpha + \beta + \delta \neq 0$ everywhere.

We now choose the vector fields L_1, L_2 and L_3 corresponding to the 1-forms α, β and δ respectively as the unit vector fields such that they are mutually orthogonal to each other. We now suppose that $\alpha(Y) \neq 0$ for all Y . For if, $\alpha(Y) = 0$ for all Y then $g(L_1, L_1) = 0$, which contradicts to our assumption that L_1 is a unit vector field. Then multiplying both sides of (4.2) by $\alpha(Y)$ we get

$$(4.3) \quad \begin{aligned} & \alpha(Y) \{\alpha(X) + \beta(X) + \delta(X)\} [S(Z, U) - \frac{r}{n}g(Z, U)] \\ & + \alpha(Y) \{\alpha(Z) + \beta(Z) + \delta(Z)\} [S(X, U) - \frac{r}{n}g(X, U)] \\ & + \alpha(Y) \{\alpha(U) + \beta(U) + \delta(U)\} [S(Z, X) - \frac{r}{n}g(Z, X)] = 0. \end{aligned}$$

Setting $X = Y = e_i$ in (4.3) and taking summation over $i, 1 \leq i \leq n$, we have

$$(4.4) \quad \begin{aligned} & S(Z, U) - \frac{r}{n}g(Z, U) + \{\alpha(Z) + \beta(Z) + \delta(Z)\} [\alpha(QU) - \frac{r}{n}\alpha(U)] \\ & + \{\alpha(U) + \beta(U) + \delta(U)\} [\alpha(QZ) - \frac{r}{n}\alpha(Z)] = 0. \end{aligned}$$

Since the manifold under consideration is of constant scalar curvature, using (2.23) in (4.4) we get

$$S(Z, U) = \frac{r}{n}g(Z, U), \quad \text{which means that the manifold is Einstein.}$$

In a similar manner multiplying (4.3) by $\beta(Y)$ and $\delta(Y)$ respectively we obtain that the manifold is Einstein. This leads to the following:

Theorem 4.1. *If in a $(WQCS)_n$, the Ricci tensor is cyclic parallel and $a + (n - 2)b \neq 0$ then it is an Einstein manifold unless $\alpha + \beta + \delta$ is non-vanishing everywhere.*

Corollary 4.1. *If a $(WQCS)_n$ is Ricci symmetric then it is an Einstein manifold, provided that $a + (n - 2)b \neq 0$ and $\alpha + \beta + \delta \neq 0$ everywhere.*

Again in [7] it is shown that if an Einstein $(WQCS)_n$ is a $(WS)_n$ then the scalar curvature of the manifold vanishes, provided that $a \neq 0$ and $\alpha + \beta + \delta \neq 0$. Hence by virtue of Theorem 4.1 we can state the following:

Theorem 4.2. *If a $(WQCS)_n$ with cyclic parallel Ricci tensor is a $(WS)_n$ then the scalar curvature of the manifold vanishes, provided that $a \neq 0$, $a + (n - 2)b \neq 0$ and $\alpha + \beta + \delta \neq 0$ everywhere.*

Next in [7] it is proved that if in an Einstein $(WQCS)_n$ the scalar curvature vanishes then it is a $(WS)_n$, provided that $a \neq 0$. Hence by virtue of Theorem 4.1 we can state the following:

Theorem 4.3. *If in a $(WQCS)_n$ with cyclic parallel Ricci tensor the scalar curvature vanishes, then it is a $(WS)_n$, provided that $a \neq 0$, $a + (n - 2)b \neq 0$ and $\alpha + \beta + \delta \neq 0$ everywhere.*

Therefore if a $(WQCS)_n$ satisfying (4.1) is of non-vanishing scalar curvature then in view of Theorem 4.3 we can state the following:

Theorem 4.4. *If in a $(WQCS)_n$ with non-vanishing scalar curvature, the Ricci tensor is cyclic parallel then it cannot be a $(WS)_n$, provided that $a \neq 0$, $a + (n - 2)b \neq 0$ and $\alpha + \beta + \delta \neq 0$ everywhere.*

Definition 4.2. A vector field L on a Riemannian manifold is said to be concurrent [6] if $\nabla_X L = \rho X$, where ρ is a constant.

In particular, if $\rho = 0$ then L is said to be a parallel vector field.

Let us now consider a $(WQCS)_n$ such that the vector field $L = L_2 + L_3$ defined by $g(X, L) = \beta(X) + \delta(X)$ is a concurrent vector field. Then making use of Ricci identity we have

$$(4.5) \quad R(X, Y, L, U) = 0 \quad \text{which implies that}$$

$$(4.6) \quad S(Y, L) = 0.$$

Now the relation (2.15) can be written as

$$(4.7) \quad S(X, L) = \frac{r}{n}g(X, L), \quad \text{provided that} \quad a + (n - 2)b \neq 0.$$

From (4.6) and (4.7) it follows that

$$r = 0, \quad \text{if} \quad \|L\|^2 \neq 0.$$

This leads to the following:

Theorem 4.5. *If in a $(WQCS)_n$ the vector field L defined by $g(X, L) = \beta(X) + \delta(X)$ is a concurrent vector field then it is of vanishing scalar curvature, provided that $a + (n - 2)b \neq 0$ and $\|L\|^2 \neq 0$.*

Since $r = 0$, from (2.20) and (2.21) we get

$$(4.8) \quad \beta(QX) = \delta(QX) = 0 \text{ if } a + (n - 2)b \neq 0.$$

Now using (4.8) and $r = 0$ in (2.14) we obtain

$$(4.9) \quad \begin{aligned} & \{a + (n - 2)b\}(\nabla_X S)(Z, U) \\ &= \{a + (n - 2)b\}[\alpha(X)S(Z, U) + \beta(Z)S(X, U) + \delta(U)S(Z, X)] \\ &+ a[\beta(R(X, Z)U) + \delta(R(X, U)Z)] + b[\beta(X)S(Z, U) \\ &- \beta(Z)S(X, U) + \delta(X)S(Z, U) - \delta(U)S(Z, X)]. \end{aligned}$$

Again from $\nabla_X L = \rho X$, we have

$$(4.10) \quad (\nabla_X S)(Z, L) = -\rho S(Z, X).$$

Setting $U = L$ in (4.9) and then using (4.5) and (4.6) we obtain by virtue of (4.10) that

$$(4.11) \quad [\rho\{a + (n - 2)b\} + \{a + (n - 3)b\}\delta(L)]S(Z, X) + a\delta(R(X, L)Z) = 0.$$

From (4.5) we have

$$\begin{aligned} R(L, U, X, Y) &= 0, \quad \text{which implies that} \\ R(U, L, Y, X) &= 0 \quad \text{for all vector fields } U, X, Y. \end{aligned}$$

The last relation yields (for $X = L_3$) that $\delta(R(U, L)Y) = 0$ for all vector fields $U, Y \in \chi(M)$. Hence $\delta(R(X, L)Z) = 0$ for all $X, Z \in \chi(M)$. Consequently (4.11) reduces to

$$S(Z, X) = 0 \quad \text{for all } X \text{ and } Z,$$

provided that $\rho\{a + (n - 2)b\} + \{a + (n - 3)b\}\delta(L) \neq 0$.

Thus (2.1) takes the form $W(X, Y, Z, U) = aR(X, Y, Z, U)$ and hence (1.5) reduces to

$$\begin{aligned} (\nabla_X R)(Y, Z, U, V) &= \alpha(X)R(Y, Z, U, V) + \beta(Y)R(X, Z, U, V) \\ &+ \beta(Z)R(Y, X, U, V) + \delta(U)R(Y, Z, X, V) \\ &+ \delta(V)R(Y, Z, U, X) \end{aligned}$$

for $a \neq 0$, which implies that the manifold is a $(WS)_n$. Thus we can state the following:

Theorem 4.6. *If in a $(WQCS)_n$ with $a \neq 0$ and $a + (n-2)b \neq 0$ the non-null vector field L defined by $g(X, L) = \beta(X) + \delta(X)$ is a concurrent vector field then it is a $(WS)_n$, provided that $\rho\{a + (n-2)b\} + \{a + (n-3)b\}\delta(L) \neq 0$.*

Corollary 4.2. *If in a $(WQCS)_n$ with $a \neq 0$ and $a + (n-2)b \neq 0$ the non-null vector field L defined by $g(X, L) = \beta(X) + \delta(X)$ is a parallel vector field then it is a $(WS)_n$, provided that $\{a + (n-3)b\}\delta(L) \neq 0$.*

The above corollary certainly improves the Theorem 4.5 of [7].

Definition 4.3. A vector field L on a Riemannian manifold is said to be *recurrent* [6] if $\nabla_X L = \mu(X)L$, where μ is a non-zero 1-form, called the associated 1-form of the recurrent vector field.

In particular, if $\mu(X)$ is a constant then the recurrent vector field reduces to a concurrent vector field.

Now we consider a $(WQCS)_n$ such that the vector field L defined by $g(X, L) = \beta(X) + \delta(X)$ is a recurrent vector field. Then we have

$$\nabla_X \nabla_Y L = (X\mu(Y))L + \mu(X)\mu(Y)L$$

and hence using Ricci identity we get

$$R(X, Y, L, U) = 2d\mu(X, Y)g(L, U) \quad \text{which implies that}$$

$$R(X, Y, L, U) = 0, \quad \text{if the 1-form } \mu \text{ is closed.}$$

Then $S(Y, L) = 0$ and hence $r = 0$. Therefore proceeding similarly as before we obtain that the manifold is a $(WS)_n$. Hence we can state the following:

Theorem 4.7. *If in a $(WQCS)_n$ with $a \neq 0$ and $a + (n-2)b \neq 0$, the vector field L defined by $g(X, L) = \beta(X) + \delta(X)$ is a recurrent vector field such that the associated 1-form of the recurrent vector field is closed then it is a $(WS)_n$, provided that $a + (n-3)b \neq 0$ and $\delta(L) \neq 0$.*

§5. Some examples of $(WQCS)_n$

This section deals with several examples of $(WQCS)_n$. We calculate the components of the curvature tensor, the Ricci tensor, the quasi-conformal curvature tensor and its covariant derivative.

EXAMPLE 1. Let $M^4 = \{(x^1, x^2, x^3, x^4) \in \mathbf{R}^4 | x^1 < 0, x^3 > 0\}$ be an open subset of \mathbf{R}^4 endowed with the metric

$$(5.1) \quad ds^2 = x^1(x^3)^2(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2.$$

Then the only non-vanishing components of the Christoffel's symbols, the curvature tensor, the Ricci tensor, the scalar curvature, the quasi-conformal curvature tensor and its covariant derivatives are

$$\begin{aligned}\Gamma_{11}^2 &= \frac{1}{2}(x^3)^2, & \Gamma_{11}^3 &= -x^1x^3 = -\Gamma_{13}^2, \\ R_{1313} &= x^1, & S_{11} &= -x^1, & r &= 0, \\ W_{1313} &= (a+b)x^1, & W_{1414} &= bx^1, \\ W_{1313,1} &= (a+b), & W_{1414,1} &= b.\end{aligned}$$

Here ‘,’ denotes the covariant differentiation with respect to the metric tensor g . Therefore our M^4 with the considered metric g in (5.1) is a Riemannian manifold of vanishing scalar curvature which is neither quasi-conformally flat nor quasi-conformally symmetric. We put

$$\begin{aligned}\alpha_i(\partial_i) &= \alpha_i = \begin{cases} \frac{1}{2x^1} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \\ \beta_i(\partial_i) &= \beta_i = \begin{cases} \frac{1}{3x^1} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \\ \delta_i(\partial_i) &= \delta_i = \begin{cases} \frac{1}{6x^1} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial x^i}$. Then (M^4, g) is a $(WQCS)_4$. Hence we can state the following:

Theorem 5.1. *Let (M^4, g) be a Riemannian manifold endowed with the metric given in (5.1). Then (M^4, g) is a weakly quasi-conformally symmetric manifold with vanishing scalar curvature which is neither quasi-conformally symmetric nor quasi-conformally recurrent.*

EXAMPLE 2. Let $M^n = \mathbf{R}^n (n \geq 4)$ be endowed with the metric

$$(5.2) \quad ds^2 = f \cdot (dx^1)^2 + \sum_{i=2}^{n-1} (dx^i)^2 + 2dx^1dx^n,$$

where f is a continuously differentiable function of x^1, x^2, \dots, x^{n-1} such that

$$(5.3) \quad f < 0, \quad af_{\cdot mmk} + b \sum_{j=2}^{n-1} f_{\cdot jjk} \neq 0 \quad \text{and} \quad af_{\cdot mm} + b \sum_{j=2}^{n-1} f_{\cdot jj} \neq 0$$

for $2 \leq m \leq n-1$ and $1 \leq k \leq n-1$ and ‘.’ denotes the partial differentiation with respect to the coordinates. Then the only non-vanishing components of

the Christoffel's symbols, the curvature tensor, the Ricci tensor, the scalar curvature, the quasi-conformal curvature tensors and their covariant derivatives are given by the following:

$$\begin{aligned}\Gamma_{11}^m &= -\Gamma_{1m}^n = -\frac{1}{2}f_{\cdot m}, & \Gamma_{11}^n &= \frac{1}{2}f_{\cdot 1}, \\ R_{1m1m} &= \frac{1}{2}f_{\cdot mm}, & S_{11} &= -\frac{1}{2}\sum_{j=2}^{n-1}f_{\cdot jj}, \quad r = 0, \\ W_{1m1m} &= \frac{1}{2}\left(af_{\cdot mm} + b\sum_{j=2}^{n-1}f_{\cdot jj}\right), \\ W_{1m1m,k} &= \frac{1}{2}\left(af_{\cdot mmk} + b\sum_{j=2}^{n-1}f_{\cdot jjk}\right).\end{aligned}$$

Thus (M^n, g) is neither quasi-conformally flat nor quasi-conformally symmetric. We set

$$\begin{aligned}\alpha_i(\partial_i) &= \alpha_i = \begin{cases} \partial_i \log |af_{\cdot mm} + b\sum_{j=2}^{n-1}f_{\cdot jj}| & \text{for } i = 1, 2, \dots, n-1 \\ 0 & \text{for } i = n, \end{cases} \\ \beta_i(\partial_i) &= \beta_i = \begin{cases} -\frac{1}{2} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \\ \delta_i(\partial_i) &= \delta_i = \begin{cases} \frac{1}{2} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial x^i}$. Then (M^n, g) is a $(WQCS)_n$. Hence we can state the following:

Theorem 5.2. *Let (M^n, g) be a Riemannian manifold equipped with the metric given in (5.2). Then (M^n, g) is a weakly quasi-conformally symmetric manifold with vanishing scalar curvature which is neither quasi-conformally symmetric nor quasi-conformally recurrent.*

EXAMPLE 3. Let $M^n = \{(x^1, x^2, \dots, x^n) \in \mathbf{R}^n | x^1 < 0, x^3 > 0\}$ be endowed with the metric

$$(5.4) \quad ds^2 = x^1(x^3)^2(dx^1)^2 + 2dx^1dx^2 + \sum_{i=3}^n(dx^i)^2.$$

Then the only non-vanishing components of the Christoffel's symbols, the curvature tensor, the Ricci tensor, the scalar curvature, the quasi-conformal

curvature tensor and their covariant derivatives are given by the following:

$$\begin{aligned}\Gamma_{11}^2 &= \frac{1}{2}(x^3)^2, & \Gamma_{11}^3 &= -x^1x^3 = -\Gamma_{13}^2, \\ R_{1313} &= x^1, & S_{11} &= -x^1, & r &= 0, \\ W_{1313} &= (a+b)x^1, & W_{1k1k} &= bx^1, \\ W_{1313,1} &= (a+b), & W_{1k1k,1} &= b\end{aligned}$$

for $4 \leq k \leq n$. We put

$$\begin{aligned}\alpha_i(\partial_i) &= \alpha_i = \begin{cases} \frac{1}{2x^1} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \\ \beta_i(\partial_i) &= \beta_i = \begin{cases} \frac{1}{3x^1} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \\ \delta_i(\partial_i) &= \delta_i = \begin{cases} \frac{1}{6x^1} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial x^i}$. Then it can be easily shown that (M^n, g) is a $(WQCS)_n$, which is neither quasi-conformally symmetric nor quasi-conformally recurrent. Hence we can state the following:

Theorem 5.3. *Let (M^n, g) ($n \geq 4$) be a Riemannian manifold equipped with the metric given in (5.4). Then (M^n, g) ($n \geq 4$) is a weakly quasi-conformally symmetric manifold with vanishing scalar curvature which is neither quasi-conformally symmetric nor quasi-conformally recurrent.*

Let (M_1^4, g_1) be a Riemannian manifold in Example 1 and (\mathbf{R}^{n-4}, g_0) be an $(n-4)$ -dimensional Euclidean space with standard metric g_0 . Then (M^n, g) in Example 3 is a product manifold of (M_1^4, g_1) and (\mathbf{R}^{n-4}, g_0) . Thus we can state the following:

Theorem 5.4. *Let (M^n, g) ($n \geq 5$) be a Riemannian manifold endowed with the metric given in (5.4). Then (M^n, g) ($n \geq 4$) is a decomposable weakly quasi-conformally symmetric manifold $(M_1^4, g_1) \times (\mathbf{R}^{n-4}, g_0)$ with vanishing scalar curvature.*

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